$boy(V) = boy(W)$ ve $L : V \to W$

1) 1-1 ise örtendir

2) Örten ise birebirdir

(1 & 2) ⇒ L nin tersi L^{-1} vardır, tektir ve lineer bir dönüşümdür.

$$boy\ çek(L) + boy\ L(V) = boy(V)$$

Örnek: $L: \mathbb{R}^2 \to \mathbb{R}^2$, $L\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

lineer dönüşümü için

a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in$ çek L midir?

$$L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \notin$ çek(L)

$$L\begin{pmatrix} 1 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

b) $\begin{bmatrix} 2 \\ -1 \end{bmatrix} \in$ çek(L) midir?

$$L\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 2 \\ -1 \end{bmatrix} \in$ çek(L) dir

c) $\begin{bmatrix} 3 \\ 6 \end{bmatrix} \in L(\mathbb{R}^2)$ midir?

d) $\begin{bmatrix} 2 \\ 3 \end{bmatrix} \in L(\mathbb{R}^2)$ midir?

öyle bir $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ var mı ki $L\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \underset{(a)}{\begin{bmatrix} 3 \\ 6 \end{bmatrix}} = \underset{(b)}{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c|c} 1 & 2 & 3 & 2 \\ 2 & 4 & 6 & 3 \end{array}\right] \xrightarrow{-2S_1+S_2} \sim \left[\begin{array}{cc|c|c} 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

$a_1 + 2a_2 = 3$

$a_1 = 3 - 2a_2$

$a_2 = 1$ için $a_1 = 1 \Rightarrow L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

(b)

$a_1 + 2a_2 = 2$

$\underline{0.a_1 + 0a_2 = -1}$

Çözüm yoktur

$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ görüntü uzayında değildir.

e) çek(L) yi bulunuz.

$$L : V \longrightarrow W$$
$$\downarrow \qquad\qquad \downarrow$$
$$\text{çek } L \qquad L(V)$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \text{çek}(L) \iff L\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right] \overset{-2S_1+S_2}{\sim} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} a_1 + 2a_2 = 0 \\ a_1 = -2a_2 \end{array}$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \text{çek}(L) \iff a_1 = -2a_2, \quad a_2 = s \text{ için} \quad a_1 = -2s$$

$$\text{çek}(L) = \left\{ \begin{bmatrix} -2s \\ s \end{bmatrix} : s \in \mathbb{R} \right\}$$

f) $L(\mathbb{R}^2)$ yi geren bir vektör kümesi bulunuz.

(i) ($L: V \to W$ verildiğinde $\{v_1, \ldots, v_k\} \subset V$ bir baz ise $L(V)$, $\{L(v_1), \ldots, L(v_k)\}$ tarafından gerilir.)

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ standart baz.

$L(\mathbb{R}^2)$; $L\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ve $L\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ tarafından gerilir.

$$L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$ $L(\mathbb{R}^2)$ yi geren.

ya da (ii) A verildiğinde $L_A = A \cdot X$, $L_A : \mathbb{R}^n \to \mathbb{R}^n$ şeklinde tanımlanan lineer dönüşümün görüntü uzayı A'nın sütun uzayıdır.

$$A \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \cdot (A_1) + x_2 (A_2) + \dots + x_n \cdot A_n$$

A nın sütunları.

$L(\mathbb{R}^2)$; A nın sutun vektörleri tarafından gerilir.

$L(R^2)$ yi geren küme $= \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$

Örnek: $L : \mathbb{R}^2 \to \mathbb{R}^3$ $\quad L\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ a_1 + a_2 \\ a_2 \end{bmatrix}$

lineer dönüşümü için;

a) $çek(L) = ?$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in çek(L) \iff L\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$L\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ a_1 + a_2 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{array}{l} a_1 = 0 \\ a_1 + a_2 = 0 \\ a_2 = 0 \end{array} \iff a_1 = a_2 = 0$$

$$çek(L) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

b) L 1-1 midir?

(Hatırlatma: L 1-1 $\Leftrightarrow$ $\text{çek}(L) = \vec{0}$)

(a)'dan $\text{çek}(L) = \{0_{\mathbb{R}^2}\}$ olduğunu gördüğümüzden,

L 1-1 dir.

c) L örten midir?

$$\underbrace{\text{boy}\ \text{çek}(L)}_{0} + \underbrace{\text{boy}\ L(\mathbb{R}^2)}_{\color{red}{2}} = \underbrace{\text{boy}(\mathbb{R}^2)}_{2}$$

$L : \mathbb{R}^2 \to \underbrace{\mathbb{R}^3}_{3 \text{ boyutlu}}$

$\text{boy}(L(\mathbb{R}^2)) = 2$

L örten değildir.

Her

Örnek: $A \in \mathbb{R}^{2\times 3}$ için $L : \mathbb{R}^{2\times 3} \longrightarrow \mathbb{R}^{3\times 3}$ $\quad \underbrace{3\times 2 \quad 2\times 3}$

$$A \longmapsto \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 3 & 1 \end{bmatrix} \cdot A$$

şeklinde tanımlanan L lineer dönüşümünün

a) çek(L)nin boyutunu bulunuz.

$$A \in \text{çek}(L) \Leftrightarrow L(A) = \underline{\begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}} \cdot A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

$2a_1 - b_1 = 0$	$a_1 + 2b_1 = 0$	$3a_1 + b_1 = 0$
$2a_2 - b_2 = 0$	$a_2 + 2b_2 = 0$	$3a_2 + b_2 = 0$
$2a_3 - b_3 = 0$	$a_3 + 2b_3 = 0$	$3a_3 + b_3 = 0$

$$
\begin{array}{cccccc|c}
a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & \\
\hline
1 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 \\
2 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & -1 & 0 \\
3 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 1 & 0
\end{array}
$$

$$
\left.
\begin{array}{cccccc|c}
1 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 \\
\hline
0 & 0 & 0 & -5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -5 & 0 \\
\hline
0 & 0 & 0 & -5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -5 & 0
\end{array}
\right\}
\begin{array}{l}
b_1 = b_2 = b_3 = 0 \\
a_1 = a_2 = a_3 = 0
\end{array}
$$

$$
\begin{array}{l|l|l}
2a_1 - b_1 = 0 & a_1 + 2b_1 = 0 & 3a_1 + b_1 = 0 \\
2a_2 - b_2 = 0 & a_2 + 2b_2 = 0 & 3a_2 + b_2 = 0 \\
2a_3 - b_3 = 0 & a_3 + 2b_3 = 0 & 3a_3 + b_3 = 0
\end{array}
$$

$$
A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \ker(L) = \{ 0^{2\times 3} \}
$$

b) $\text{boy}\, L(\mathbb{R}^{2\times 3}) = ?$ (6)

$$\underbrace{\text{boy çek}(L)}_{0} + \underbrace{\text{boy}\, L(\mathbb{R}^{2\times 3})}_{6} = \underbrace{\text{boy}(\mathbb{R}^{2\times 3})}_{2\cdot 3 = 6}$$

Örnek: $L: P_2 \to P_1$, $L(at^2+bt+c) = (a+b).t + (b-c)$

lineer dönüşümü için;

a) çek(L) nin bir bazını bulunuz.

$at^2+bt+c \in \text{çek}(L) \Leftrightarrow L(at^2+bt+c) = (a+b).t + (b-c) = 0$

$$\begin{array}{l} a+b=0 \\ b-c=0 \end{array} \Rightarrow \begin{array}{ccc} a & b & c \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$b=c$, $c=s$ için

$b=s$

$a=-b=-s$

$$\text{çek}(L) = \{ -s.t^2 + s.t + s : s \in \mathbb{R} \}$$
$$= \{ s.\underbrace{(-t^2+t+1)} : s \in \mathbb{R} \}$$

çek(L) nin bir bazı $= \{ -t^2+t+1 \}$

boy çek(L) $= \underline{1}$

b) $L(P_2)$ için bir baz bulunuz.

(i) $L(P_2) = \{ (a+b).t + b - c \;:\; at^2+bt+c \in P_2 \}$

$$(a+b).t + (b-c) = \underset{a_1}{a} \cdot t + \underset{a_2}{b}(t+1) - \underset{a_3}{c} \cdot 1$$

$\Rightarrow \{t, \cancel{t+1}, 1\}$ $L(P_2)$'yi geren bir vektör kümesi

Lin. Bğ.sız

$\Rightarrow \{t, 1\}$ $L(P_2)$ için bir bazdır.

(ii) P_2 nin bir bazı $\{t^2, t, 1\}$

$$\left.\begin{aligned} L(t^2) &= t \\ L(t) &= t+1 \\ L(1) &= -1 \end{aligned}\right\} \Rightarrow L(P_2), \ \{t, \ t+1, \ (-1)\}$$

tarafından gerilir.

$\{t, -1\}$ aradığımız bazdır.

Örnek: $L : \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^{2\times 2}$, $L(A) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \cdot A - A \cdot \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

lin. dönüşümü veriliyor;

a) çek(L) nin bir bazını bulunuz.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{çek}(L) \Longleftrightarrow L(A) = \underbrace{\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}}$$

$$L(A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a+2c & b+2d \\ a+c & b+d \end{bmatrix} - \begin{bmatrix} a+b & 2a+b \\ c+d & 2c+d \end{bmatrix} = \begin{bmatrix} 2c-b & 2d-2a \\ a-d & b-2c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$a - d = 0$

$b - 2c = 0$

$2c - b = 0$

$2d - 2a = 0$

$$\begin{array}{cccc|c} a & b & c & d & \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 \\ -2 & 0 & 0 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$b = 2c$
$a = d$

$$\text{çek}(L) = \left\{ \begin{bmatrix} d & 2c \\ c & d \end{bmatrix} : c, d \in \mathbb{R} \right\}$$

$$a - d = 0$$
$$b - 2c = 0$$
$$2c - b = 0$$
$$2d - 2a = 0$$

$$\begin{bmatrix} d & 2c \\ c & d \end{bmatrix} = d . \underline{\underline{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}} + c \underline{\underline{\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}}}$$

$$\text{çek}(L) \text{ nin bir bazı} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \right\}$$

b) $L(\mathbb{R}^{2\times 2})$ nin bir bazını bulunuz.

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ için } \quad L(A)=\left\{\begin{bmatrix} 2c-b & 2d-2a \\ a-d & b-2c \end{bmatrix} : a,b,c,d\in\mathbb{R}\right\}$$

$$\begin{bmatrix} 2c-b & 2d-2a \\ a-d & b-2c \end{bmatrix} = c\cdot\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} + b\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + d\cdot\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$$

$$+\,a\cdot\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$$

$\left\{\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}\right\}$ $L(\mathbb{R}^{2\times 2})$ yi gerer.

iki uzay izomorftur $\Leftrightarrow$ boyutları eşit

$$\left.\begin{array}{lll} \mathbb{R}^{2\times 2} & 4 & \text{boyutlu} \\ \mathbb{R}^{4} & 4 & \text{boyutlu} \end{array}\right\} \quad \mathbb{R}^{2\times 2} \simeq \mathbb{R}^{4}$$

$$\left\{ \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \right\}$$

$$\begin{bmatrix} x & y \\ z & t \end{bmatrix} \longmapsto \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \qquad \boxed{\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -1 & 1 \\ -2 & 1 & 0 & 0 \end{bmatrix}} \rightarrow$$

Lin. Bağlılık Alg. uygulanır.

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1. 3.

$$\text{Baz} = \left\{ \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 2 & 0 & 0 & -2 \\ -1 & 0 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & -2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 2 & 0 & 0 & -2 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \; > \; \{(1,0,0,-1),(0,2,-1,0)\}$$

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} \right\} = Baz_2$$